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Nonhomogeneous Boundary Value Problems

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The diffusion equation is solved under stochastic nonhomogeneity using eigen function expansion and the Georges method. The statistical moments of the solution process are computed through the two previously mentioned techniques and proved to be the same. A general solution is obtained under general initial and boundary conditions. A random source composed of deterministic and stochastic parts is taken into consideration. The stochastic part is then restricted to a generalized Gaussian field, mainly modulated white noise. A special case is considered under constant noise level and constant average noise. A numerical case study concerning pollution in a stream is solved and a parametric study is achieved through various figures. © 1996 Academic Press, Inc.

1. INTRODUCTION

In this investigation, the nonhomogeneous stochastic diffusion equation is solved. The homogeneous stochastic diffusion equation was the concern of the author and others in a lot of publications; for example, see [1–4]. It is well known that these equations are essential in many engineering and scientific applications, for example; heat conduction; concentration diffusion problems, and phase transitions using white noise disturbance as in [5].

A random source is assumed which has a deterministic ensemble average. A stochastic component with zero mean is added for which a random solution exists. The statistical moments of the solution are computed using two independent techniques; the eigen function expansion [6] and Georges method [7]. The two different techniques produce similar statistical moments. However, the eigen function expansion seems to be familiar with those whom partial differential equations are their interests, but Georges method is not. Accordingly, Georges method is summarised in Appendix A after some simplifications to fit the problem in hand.

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A simple but important application is studied to illustrate the obtained results in which a stream is subjected randomly to pollution throughout its length and in its boundary and initial conditions. Significant results are obtained and illustrated through figures.

2. PROBLEM FORMULATION

The stochastic parabolic equation with diffusion coefficient α^2 and random source $f(x, t; \omega)$ is given by

$$u_t(x, t; \omega) = \alpha^2 u_{xx}(x, t; \omega) + f(x, t; \omega), \tag{1}$$

$(x, t) \in (0, L) \times (0, \infty)$ and $\omega \in (\Omega, B, P)$; a complete probability space.

The boundary conditions are generally functions of time in the forms $u(0, t) = g_1(t)$ and $u(L, t) = g_2(t)$, $t \in [0, \infty)$. The initial condition is assumed to be in the form $\text{Lim}_{t \to 0} u(x, t; \omega) = \phi(x)$, $x \in [0, L]$. The random source is assumed as the expression

$$f(x, t; \omega) = G(x, t) + e(x, t) n(t; \omega), \tag{2}$$

where $G(x, t)$ is a deterministic function which is the ensemble average of f and $n(t; \omega)$ is a generalized Gaussian random field with extremely short correlations or white noise which has the statistical moments

$$En(t; \omega) = 0, \qquad En(t_1; \omega) \cdot n(t_2; \omega) = \delta(t_1 - t_2),$$

in which $\delta(\cdot)$ is Dirac delta function. The function $e(x, t)$ is an envelope function for the modulated white noise. We suppose that f is continuous over $D = (0, L) \times (0, \infty)$, $\phi(x)$ is continuous over $[0, L]$ and both $g_1(t)$ and $g_2(t)$ are differentiables over $[0, \infty)$.

3. SOLUTION USING EIGEN FUNCTION EXPANSION

In this technique the solution is assumed to take the form

$$u(x, t; \omega) = \sum_{n=1}^{\infty} A_n(x) \cdot B_n(t; w). \tag{3}$$

Executing the algorithm of eigen function expansion [6], the results

$$u(x, t; \omega) = u_s(x, t) + \sum_{n=1}^{\infty} a_n \exp(n, t) \cdot Si(n, x) + \sum_{n=1}^{\infty} z(n, t) \cdot Si(n, x) \tag{4}$$

are obtained, where

$$a_n = (2/L)\int_0^L (\phi(x) - u_s(x,0)) \cdot Si(n,x)\,dx, \tag{5}$$

$$z(n,t) = \int_0^L \exp(n, t-\tau) f_n(\tau;\omega)\,d\tau \tag{6}$$

in which

$$f_n(t;\omega) = (2/L)\int_0^L \big(f(x,t;\omega) - u_{s_t}(x,t)\big) \cdot Si(n,x)\,dx, \tag{7}$$

$\exp(n,t) = e^{-(n\pi\alpha/L)^2 \cdot t}$, and $Si(n,x) = \sin(n\pi x/L)$. The term $u_s(x,t)$ takes the form

$$u_s(x,t) = g_1(t)(1 - x/L) + g_2(t)(x/L). \tag{8}$$

Computing the ensemble average and variance of $u(x,t;\omega)$, we obtain

$$\begin{aligned} \mathbf{E}u(x,t;\omega) &\\ &= u_s(x,t) + \sum_{n=1}^{\infty} a_n \exp(n,t) \cdot Si(n,x) \\ &\quad + (2/L)\sum_{n=1}^{\infty}\left(\int_0^t \exp(n,t-\tau)\left(\int_0^L G(x,\tau) Si(n,x)\,dx\right)d\tau\right) \\ &\quad\quad \cdot Si(n,x) \\ &\quad - (2/L)\sum_{n=1}^{\infty}\left(\int_0^t \exp(n,t-\tau)\left(\int_0^L u_{s_\tau}(x,\tau) Si(n,x)\,dx\right)d\tau\right) \\ &\quad\quad \cdot Si(n,x), \end{aligned} \tag{9}$$

$$\mathrm{VAR}\, u(x,t;\omega) = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \mathrm{COV}(M_k, M_j) Si(j,x) \cdot Si(k,x), \tag{10}$$

where

$$M_n(t;\omega) = \int_0^t \exp(n,t-\tau) \cdot f_n(\tau;\omega)\,d\tau. \tag{11}$$

The covariance term in equality (10) is computed as

$$
\begin{aligned}
&(L^2/4)\mathrm{COV}(M_k, M_j)\\
&\quad = \int_0^t \int_0^t \exp(j, t-\zeta)\cdot \exp(k, t-\tau)\\
&\qquad \cdot \left(\int_0^L \int_0^L G(x_1,\zeta)G(x_2,\tau)\cdot Si(j,x_1)\cdot Si(k,x_2)\,dx_1\,dx_2 \right) d\zeta\, d\tau\\
&\qquad + \int_0^t \exp(j, t-\tau)\exp(k, t-\tau)\\
&\qquad \cdot \left(\int_0^L \int_0^L e(x_1,\tau)\cdot e(x_2,\tau)\cdot Si(j,x_1)\cdot Si(k,x_2)\,dx_1\,dx_2 \right) d\tau\\
&\qquad - \int_0^t \int_0^t \exp(j, t-\zeta)\cdot \exp(k, t-\tau)\\
&\qquad \cdot \Bigg(\int_0^L \int_0^L \big(G(x_1,\zeta)\cdot g_{1_\tau}(1-x_2/L)\\
&\qquad + G(x_1,\zeta)\cdot g_{2_\tau}(x_2/L)\big)\cdot Si(j,x_1)\cdot Si(k,x_2)\,dx_1\,dx_2 \Bigg) d\zeta\, d\tau\\
&\qquad - \int_0^t \int_0^t \exp(j, t-\zeta)\cdot \exp(k, t-\tau)\\
&\qquad \cdot \Bigg(\int_0^L \int_0^L \big(G(x_2,\tau)\cdot g_{1_\zeta}(1-x_1/L) + G(x_2,\tau)\cdot g_{2_\zeta}(x_1/L)\big)\\
&\qquad \cdot Si(j,x_1)\cdot Si(k,x_2)\,dx_1\,dx_2 \Bigg) d\zeta\, d\tau\\
&\qquad + \int_0^t \int_0^t \exp(j, t-\zeta)\cdot \exp(k, t-\tau)\\
&\qquad \cdot \left(\int_0^L \int_0^L u_{s_\zeta}(x_1,\zeta)\cdot u_{s_\tau}(x_2,\tau)\cdot Si(j,x_1)\cdot Si(k,x_2)\,dx_1\,dx_2 \right) d\zeta\, d\tau\\
&\qquad - \mathbf{E}M_j \cdot \mathbf{E}M_k,
\end{aligned}
\tag{12}
$$

where

$$
\begin{aligned}
\mathbf{E}M_n = (2/L)\int_0^t \exp(n, t-\zeta)&\\
\cdot \left(\int_0^L \big(G(x,\zeta) - u_{s_\zeta}(x,\zeta)\big) Si(n,x)\,dx \right) d\zeta&.
\end{aligned}
\tag{13}
$$

4. SOLUTION USING GEORGES METHOD

Georges [7] introduced a random generalized solution to heat equations and applied his interesting results on some applications. In view of Appendix A, the problem in this paper has the following data:

$V = (0, L)$, opened set in the usual topology $\mathbb{R}$,

$\partial V = \{0, L\}$, a boundary set in $\mathbb{R}$,

$\overline{V} = V \cup \partial V = [0, L]$, a closure of V in $\mathbb{R}$,

$f_1(x, t; \omega) = f(x, t; \omega) - u_{s_t}(x, t)$, a continuous function over D,

$f_2(x; \omega) = \Phi(x) = \phi(x) - u_s(x, 0)$, $x \in \overline{V}$, a continuous function over $\overline{V}$.

The Green's function [7] related to the given problem is

$$\Gamma(x, t; \zeta, \tau) = (2/L) \sum_{n=1}^{\infty} \exp(n, t - \tau) Si(n, x) . Si(n, \zeta).$$

Applying these data in the generalised solution (see Appendix A), we obtain

$$(L/2) \cdot u(x, t; \omega) = \sum_{n=1}^{\infty} \exp(n, t) Si(n, x) \int_0^t \exp(n, -\tau) \int_0^L f_1(\zeta, \tau; \omega) .$$

$$Si(n, \zeta)\, d\zeta\, d\tau + \int_0^L \Phi(\zeta) \sum_{n=1}^{\infty} \exp(n, t) Si(n, x) . Si(n, \zeta)\, d\zeta. \quad (14)$$

It can be simply shown that equality (14) is equivalent to equality (4). Consequently, the ensemble average and variance of u should be the same. The expressions of the average and variance of u in Appendix A prove that they produce the same expressions obtained by the eigen function expansion technique.

5. A SPECIAL CASE

Let us compute the expressions in Eqs. (9) and (10) under the following assumptions:

1. *time invariant source average,*

$$G(x, t) = G(x); \quad (15)$$

2. *constant noise envelope,*

$$e(x, t) = K. \quad (16)$$

Performing the necessary computations using the previous assumptions in considerations, the ensemble average and variance of u have the expressions, respectively,

$$\begin{aligned}\mathbf{E}u(x,t;\omega) = u_s(x,t) &+ \sum_{n=1}^{\infty} a_n \exp(n,t)\cdot Si(n,x)\\ &+\left(2L/\alpha^2/\pi^2\right)\sum_{n=1}^{\infty}(1-\exp(n,t))/n^2\\ &\times\left(\int_0^L G(x)Si(n,x)\,dx\right)\cdot Si(n,x)\\ &-(2/\pi)\sum_{n=1}^{\infty}\exp(n,t)\cdot\Theta_n(t)\cdot Si(n,x),\end{aligned} \tag{17}$$

where

$$\Theta_n(t) = \frac{1}{n}\int_0^t \exp(n,-\tau)\cdot g_{1_\tau}\,d\tau - \frac{(-1)^n}{n}\int_0^t \exp(n,-\tau)g_{2_\tau}\,d\tau_\tau, \tag{18}$$

and

$$\begin{aligned}&\mathrm{COV}\left(M_j,M_k\right)\\ &\quad= K_1\left(\int_0^L G(x)\cdot Si(j,x)\,dx\right)\left(\int_0^L G(x)\cdot Si(k,x)\,dx\right)(1-\exp(j,t))\\ &\qquad\times(1-\exp(k,t)) + K_2(1-\exp(Q,t)) - K_3\exp(k,t)\\ &\qquad\times\left(\int_0^L G(x)\cdot Si(j,x)\,dx\right)(1-\exp(j,t))\cdot\left(k\Theta_k(t)\right)\\ &\qquad- K_4\exp(j,t)\left(\int_0^L G(x)\cdot Si(k,x)\,dx\right)(1-\exp(k,t))\\ &\qquad\cdot\left(j\Theta_j(t)\right) + K_5\exp(Q,t)jk\Theta_k(t)\Theta(t)\\ &\qquad- K_6\exp(Q,t)I_j(t)\cdot I_k(t),\end{aligned} \tag{19}$$

where

$$K_1 = 4L^2/j^2/k^2/\pi^4/\alpha^4,$$

$$K_2 = \left(4K^2\left(1-(-1)^j\right)\cdot\left(1-(-1)^k\right)L^2/jk/\left(j^2+k^2\right)/\pi^4/\alpha^2,\right.$$

$$K_3 = 4L/k/j^2/\pi^3/\alpha^2, \qquad K_4 = 4L/j/k^2/\pi^3/\alpha^2, \qquad K_5 = 4/jk/\pi^2,$$

$$K_6 = 4/L^2, \qquad Q = \sqrt{j^2+k^2},$$

and

$$I_n(t) = \int_0^t \exp(n, -\tau) \int_0^L G(x) Si(n, x)\, dx$$
$$+ (L/n/\pi)\left(-g_{1_\tau} + (-1)^n g_{2_\tau}\right) d\tau.$$

6. APPLICATION

Let us suppose that a pollutant is being carried along in a stream. The concentration of the substance, namely $u(x, t; \omega)$, changes as a function of the stream axis x and time t. The convection is neglected and accordingly the rate of change of the concentration u_t is measured by the diffusion equation (1).

Let us also suppose that the initial condition is the Dirac delta function $\delta(x)$, i.e., a pulse of pollution at the near end of the stream. A random source of pollution is assumed to exist all over he stream body with constant mean F and stochastic part $Kn(t; \omega)$. A constant level of pollution, M_1, is proposed in the near end while M_2 is the level of pollution of the far end.

Performing the necessary (and very lengthy) computations in Eqs. (17) and (19), the following results are obtained:

$$\begin{aligned}\mathbf{E}u(x, t; \omega) &= u_s(LZ, T) + (2/\pi) \\ &\times \sum_{n=1}^{\infty} \left(\left(-M_1 + (-1)^n M_2\right)/n\right) \exp(n, T) \cdot Si(n, LZ) \\ &+ \left(2/\pi^3\right) M_3 \sum_{n=1}^{\infty} \left(\left(1 - (-1)^n\right)/n^3\right)(1 - \exp(n, T)) \\ &\cdot Si(n, LZ),\end{aligned} \tag{20}$$

where $T = \alpha^2 t/L^2$, $Z = x/L$, and $M_3 = FL^2/\alpha^2$, and

$$\alpha^2 \,\mathrm{VAR}\, u(x, t; \omega)/K^2/L^2 = \left(4/\pi^4\right) \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} K_7 (1 - \exp(Q, T)) \cdot Si(j, LZ) Si(k, LZ), \tag{21}$$

where

$$K_7 = \left(1 - (-1)^j\right)\left(1 - (-1)^k\right)/jk/Q^2.$$

The root mean square error, σ, of the concentration is computed as the nonnegative square root of the variance.

6.1. *Numerical Example*

For illustration, numerical values are selected for different governing parameters of the previous application. A computer FORTRAN 77 program is achieved and the following results are obtained: Figures 1a, b illustrate the case of zero near and far end pollutions. Figures 2a, b

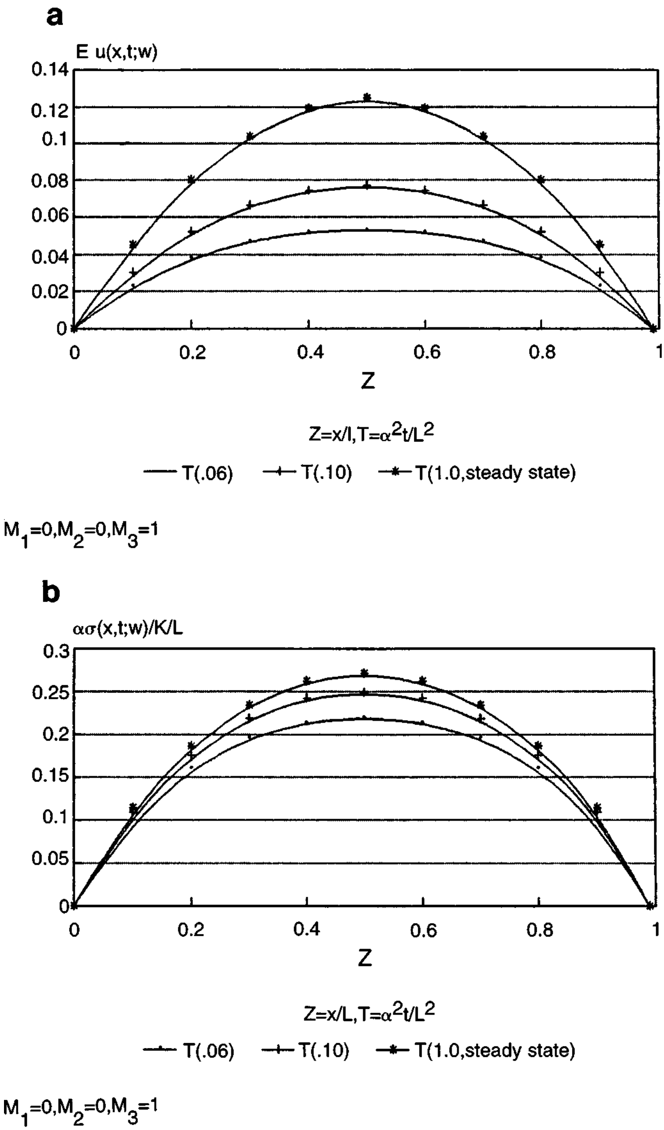

FIG. 1. (a) The average of u versus Z, T. (b) The root m.s. of u versus Z, T.

illustrate the case of only far end pollution, while Figs. 3a, b illustrate the reverse logic. Figures 1–3 are plotted at the same average noise pollution level, namely $M_3 = 1$. In plotting Fig. 4, M_3 are changed, using equal pollutions at the ends. Figure 5 illustrates the effect of changing the random noise level, K.

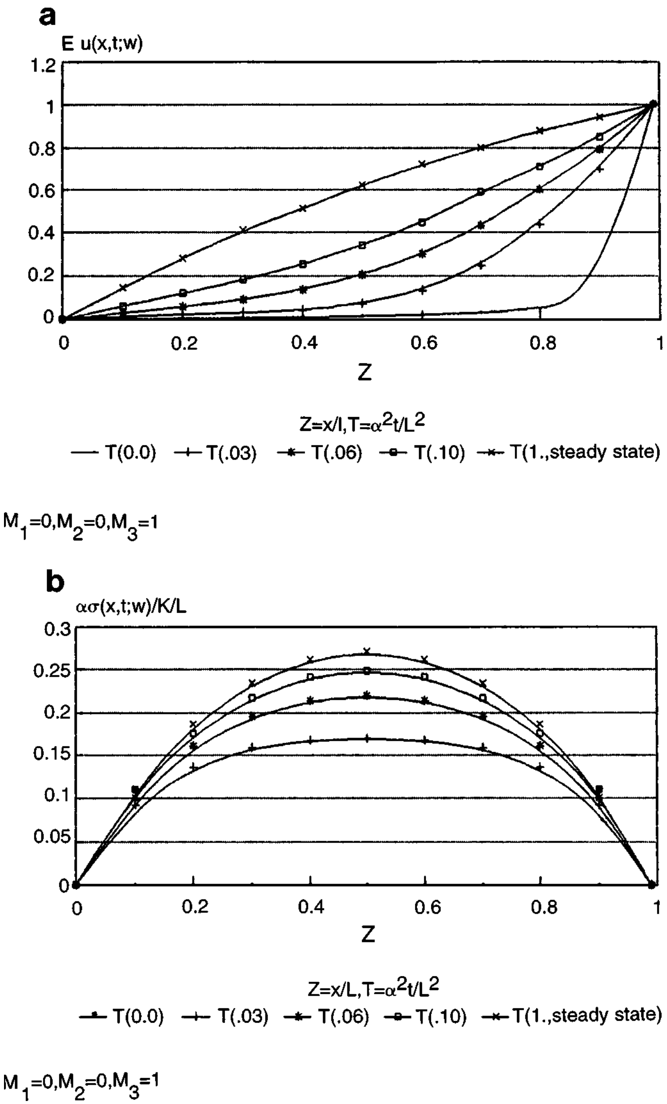

FIG. 2. (a) The average of u versus Z, T. (b) The root m.s. of u versus Z, T.

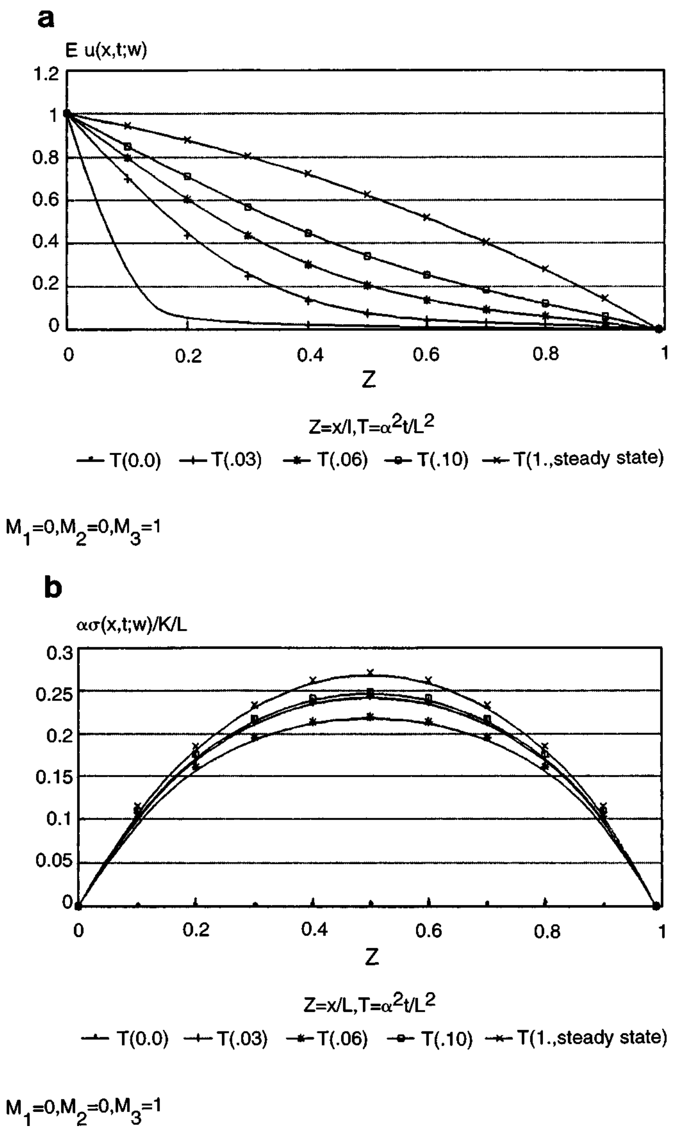

FIG. 3. (a) The average of u versus Z, T. (b) The root m.s. of u versus Z, T.

6.2. *Conclusions and Results*

The following results are extracted from studying the figures.

1. The plot of the root mean square should not be neglected since its level is comparable with the average value or even higher, as shown in Figs. 1.

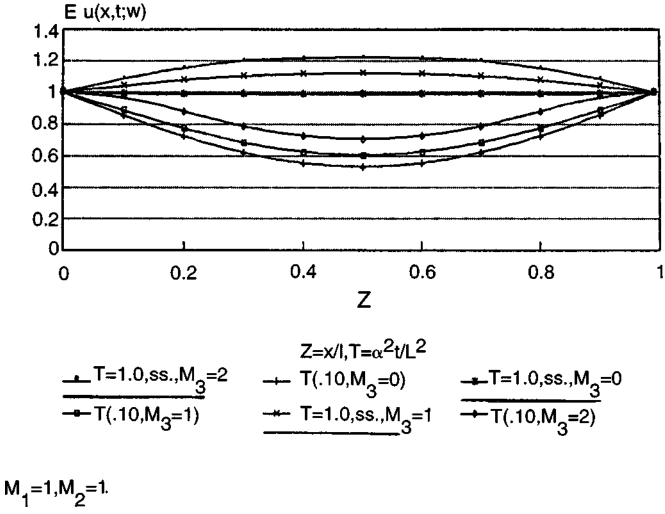

FIG. 4. The average of u versus Z, T.

2. The maximum error always occurs at the stream midlength. Accordingly, we expect the pollution concentration uncertainity to be maximum in the middle area of the stream.

3. The increase of M_3 (the increase of F) increases the average pollution probably beyond the boundary levels, as shown in Fig. 4. The root mean square error is not affected by F.

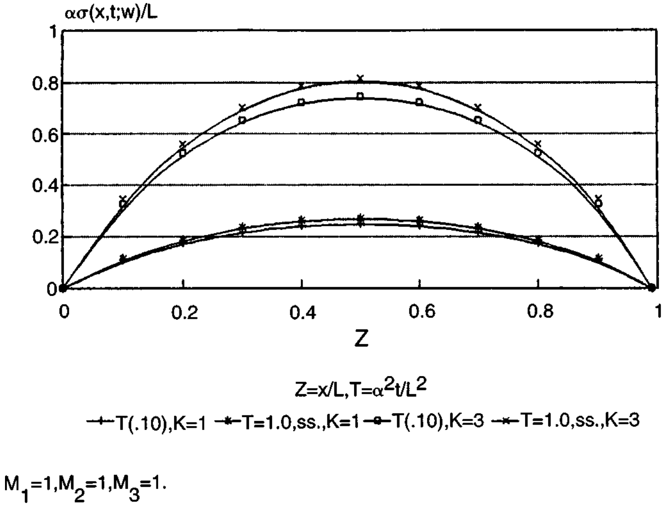

FIG. 5. The root of m.s. of u versus Z, T.

4. The increase of K, the random noise level, increases the root mean square error as shown in Fig. 5. The average of u is not affected by K.

7. GENERAL CONCLUSIONS

1. We cannot neglect random uncertainity in diffusion problems with random nonhomogeneity.

2. In the case of a random source composed of a deterministic function as its average and a modulated white noise or a zero mean Gaussian process, the average level greatly affects the solution process average while the random noise level only affects the root mean square of the solution.

3. Due to the problem linearity, the solution process is a Gaussian process with mean $\mathbf{E}u$ and variance $\mathrm{VAR}\, u$ for every point (x,t) if the source process is a Gaussian process.

APPENDIX A

Consider the problem

$$u_t = \nabla^2 u + f_1(x,t;\omega), \qquad (x,t) \in D = V \times (0,\infty), \qquad \text{(A.1)}$$

where $\omega \in (\Omega, B, P)$ is a complete probability space. The initial condition is such that $u(x,0) = f_2(x)$, $x \in \overline{V}$; the closure of V. The boundary conditions are given as

$$\operatorname*{Lim}_{\eta \to x} u(\eta, t) = f_3(x,t), \qquad x \in \partial V; \text{the boundary of } V,\ t \in [0,\infty).$$

The set V is an open region in $\mathbb{R}^n$. The functions f_1, f_2, and f_3 are continuous functions over D, $\overline{V}$, and $\partial V \times [0,\infty)$, respectively.

The solution can be written in terms of the causal Green's function $\Gamma(x,t;\kappa,\tau)$ as

$$\begin{aligned} u(x,t) = {} & \int_0^t \int_V f_1(\kappa,\tau)\Gamma(x,t;\kappa,\tau)\, d\kappa\, d\tau \\ & + \int_V f_2(\kappa)\Gamma(x,t;\kappa,0)\, d\kappa \\ & - \int_0^t \int_{\partial V} f_3(\kappa,\tau)\,\partial\Gamma(x,t;\kappa,\tau)/\partial n \cdot d\sigma(\kappa)\, d\tau \end{aligned} \qquad \text{(A.2)}$$

with n as the outward normal to ∂V and $\sigma(\kappa)$ as a function of κ. Equation (A.2) can be simplified as

$$u(x,t) = H[f_1, f_2, f_3, x, t]. \tag{A.3}$$

For homogeneous boundary conditions, f_3 is zero and the solution is reduced to

$$u(x,t) = H[f_1, f_2, x, t]. \tag{A.4}$$

THEOREM. *The expectation*, $\mathbf{E}u$, *of the solution is given by*

$$\mathbf{E}u = H[\mathbf{E}f_1, \mathbf{E}f_2, x, t]. \tag{A.5}$$

The proof of the theorem is in [7].

THEOREM. *Let* $R_{ij} = \mathbf{E}f_i f_j$, $i, j = 1, 2$, *denote the correlation functions of the* f_i*'s. The correlation function* $R_u(x, t; \kappa, \tau)$ *of* u *satisfies*

$$R_u(x,t;\kappa,\tau) = H[H[R_{11}, R_{12}, \kappa, \tau], H[R_{12}, R_{22}, \kappa, \tau], x, t]. \tag{A.6}$$

The proof of the theorem is in [7].

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